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ON COMPLETING UNIMODULAR POLYNOMIAL VECTORS OF LENGTH THREE

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ABSTRACT. It is shown that if R is a local ring of dimension three, with $\frac{1}{2} \in R$, then a polynomial three vector $(v_0(X), v_1(X), v_2(X))$ over $R[X]$ can be completed to an invertible matrix if and only if it is unimodular. In particular, if $1/3! \in R$, then every stably free projective $R[X_1, \ldots, X_n]$-module is free.

1. INTRODUCTION

In [6] A. Suslin queries

A. Suslin's question $(S_r(R))$. *Let R be a local ring. If $1/r! \in R$, can every unimodular $(r+1)$-vector over $R[X]$ be completed to an invertible matrix?*

In this note we settle $S_r(R)$ when R is a noetherian local ring of Krull dimension three.

Let us briefly recapitulate known results on $S_r(R)$. Let R be a *two* dimensional noetherian local ring. A beautiful theorem of L. N. Vaserstein in [8] identifies the set $\mathrm{Um}_3(R[X])/E_3(R[X])$ with the Elementary Symplectic Witt group $\mathrm{W_E}(R[X])$. If $1/2 \in R$, a well-known theorem of M. Karoubi asserts that any invertible alternating matrix over a polynomial ring $R[X]$ is stably congruent to its constant form. In particular, the Symplectic Witt group $\mathrm{W}(R[X]) \equiv 0$. M. P. Murthy had remarked that these two facts could be used to prove that every $v \in \mathrm{Um}_3(R[X])$ can be completed to an invertible matrix. We expanded on this theme of M. P. Murthy, in [3], to show that $S_d(R)$ holds. Here we extend the methods in [3] to prove

Theorem. *Let R be a noetherian, local ring of Krull dimension three with $1/2 \in R$. Then every unimodular 3-vector over $R[X]$ can be completed to an invertible matrix.*

The reader can also find some very interesting results on A. Suslin's question, due to M. Roitman, in positive prime characteristics in [5]. The present approach had its genesis in [2], (of course, with roots in Vaserstein theory developed in [8], and guided by M. P. Murthy's remark), where I could extend some of M. Roitman's results in dimensions ≤ 4.

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2. Preliminary remarks and calculations

All rings A considered in this article will be commutative with an identity element and noetherian. A vector $v = (v_0, v_1, \dots, v_r) \in A^{r+1}$ is said to be *unimodular* if there is a vector $w = (w_0, w_1, \dots, w_r) \in A^{r+1}$ such that $v_0 w_0 + \cdots + v_r w_r = 1$. $\mathrm{Um}_{r+1}(A)$ will denote the set of all unimodular vectors $v \in A^{r+1}$. The group $Gl_{r+1}(A)$ of invertible matrices acts on A^{r+1} in a natural way: if $v \in A^{r+1}$, $\sigma \in Gl_{r+1}(A)$ then σ will map v to $v\sigma$. Under this action $\mathrm{Um}_{r+1}(A)$ is mapped onto itself; and so $Gl_{r+1}(A)$ acts on $\mathrm{Um}_{r+1}(A)$. We let $\sim$ denote equivalence of two vectors under this action. Let $E_{r+1}(A)$ denote the subgroup of $Gl_{r+1}(A)$ consisting of all the elementary matrices, i.e. those matrices which are a finite product of matrices of the form $E_{ij}(\lambda)$, $i \neq j$, $\lambda \in A$, which has all its diagonal entries one, has one off-diagonal entry in the (i, j)th position equal λ, and has all other entries zero. $v \underset{E}{\sim} w$ will denote that v can be elementarily transformed to w. Let $\mathrm{Um}_{r+1}(A)/E_{r+1}(A)$ be the set of equivalence classes of vectors v under the equivalence $\underset{E}{\sim}$ induce by the action of $E_{r+1}(A)$ on $\mathrm{Um}_{r+1}(A)$; and let $[v]$ denote the equivalence class of $v \in \mathrm{Um}_{r+1}(A)$ in $\mathrm{Um}_{r+1}(A)/E_{r+1}(A)$.

(2.1) **W. Van der Kallen's group structure on** $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$**.** If A is a ring whose maximal spectrum $\mathrm{Max}(A)$ is a finite union of subsets V_i where each V_i, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension $\leq d$ we shall say that A is *essentially of dimension* d. For instance, a ring of Krull dimension d is obviously essentially of dimension $\leq d$; a local ring of dimension d is essentially of dimension 0; whereas a polynomial extension $R[X]$ of a local ring R of dimension $d \geq 1$ has dimension $d+1$ but is essentially of dimension d as $\mathrm{Max}(R[X]) = \mathrm{Max}(R/(a)[X]) \cup \mathrm{Max}(R_a[X])$ for any non-zero-divisor $a \in R$.

In [9, Theorem 3.6], W. Van der Kallen has described how one could have an abelian group structure on $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$. In the sequel we shall always refer to this group structure on $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$; and let $*$ denote the group multiplication henceforth. One has

(2.1.1) *Remark.* Let A be essentially of dimension $d \geq 2$, and let $C_{d+1}(A)$ denote the set of all completable $(d+1)$-vectors in $\mathrm{Um}_{d+1}(A)$. Then,

- (i) The map $\sigma \to [e_1\sigma]$, where $e_1 = (1, 0, \dots, 0) \in \mathrm{Um}_{d+1}(A)$, is a group homomorphism $Sl_{d+1}(A) \to \mathrm{Um}_{d+1}(A)/E_{d+1}(A)$.
- (ii) $C_{d+1}(A)/E_{d+1}(A)$ is a subgroup of $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$.

Proof. (i) follows from [9, Theorem 3.16(iv)]. Since any $v \in C_{d+1}(A)$ can be completed to a matrix of determinant one, $C_{d+1}(A)/E_{d+1}(A)$ is the image of $Sl_{d+1}(A)$ under the homomorphism mentioned in (i); whence it is a subgroup of $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$.

(2.2) **On A. Suslin's procedure for completing** $(a_0, a_1, a_2^2, \dots, a_r^r)$. In [6, Proposition 1.6] A. Suslin shows that if $(a_0, a_1, \dots, a_r) \in \mathrm{Um}_{r+1}(A)$ then $(a_0, a_1, a_2^2, \dots, a_r^r)$ can be completed. His proof, as observed by M. P. Murthy in [1, Chapter V, Proposition 1.2], actually demonstrates,

(2.2.1) **Proposition.** *Let* $(a_0, a_1, \dots, a_r) \in \mathrm{Um}_{r-1}(A)$. *Suppose that* $(\overline{a}_0, \overline{a}_1, \dots, \overline{a}_{r+1})$ *is completable in* $\overline{A} = A/(a_r)$. *Then* $(a_0, a_1, \dots, a_r^r)$ *is completable.*

As an application of this proposition we have

(2.2.2) **Proposition.** *Let* R *be a local ring of dimension* 3 *with* $1/2 \in R$. *Let* $v = (v_0, v_1, v_2, v_3) \in \mathrm{Um}_4(R[X])$. *Then* v *is completable if and only if* $v^{(2)} = (v_0^2, v_1, v_2, v_3)$ *is completable.*

Proof. By [3, Example 1.5.3 and Lemma 1.3.1],

$$[v^{(2)}] = [v] * [v]$$

in $\mathrm{Um}_4(R[X])/E_4(R[X])$. By Remark 2.1.1, v is completable implies that $v^{(2)}$ is also completable.

Conversely, let $v^{(2)}$ be completable. By [3, Proposition 1.4.4],

$$v \underset{E}{\sim} (w_0, w_1, w_2, c)$$

with $c \in R$ a non-zero-divisor. As mentioned in the introduction (or cf. [3, Theorem 2.5]), since $\dim R/(c) = 2$ and $1/2 \in R$,

$$(\overline{w}_0, \overline{w}_1, \overline{w}_2) \in e_1 Sl_3(R/(c)[X]).$$

By Proposition 2.2.1, (w_0, w_1, w_2, c^3) is completable. Thus,

(i) $(v_0, v_1, v_2, v_3^3) \underset{E}{\sim} (w_0, w_1, w_2, c^3)$ by [10, Theorem],

(ii) $[v]^n = [(v_0, v_1, v_2, v_3^n)]$ for all n by [3, Example 1.5.3 and Lemma 1.3.1].

Hence $[v]^2 = [v^{(2)}] \in C_4(R[X])/E_4(R[X])$, and $[v]^3 = [(w_0, w_1, w_2, c^3)] \in C_4(R[X])/E_4(R[X])$. By Remark 2.1.1, $[v] \in C_4(R[X])/E_4(R[X])$, i.e. v is completable.

(2.3) **The elementary symplectic Witt group** $\mathrm{W_E}(A)$. If $\alpha \in M_r(A)$, $\beta \in M_s(A)$ are matrices then $\alpha \perp \beta$ denotes the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \in M_{r+s}(A)$. ψ_1 will denote $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in E_2(\mathbf{Z})$, and ψ_r is inductively defined by $\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(\mathbf{Z})$, for $r \geq 2$.

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If $\varphi \in M_{2r}(A)$ is alternating then $\det(\varphi) = (\mathrm{pf}(\varphi))^2$ where pf is a polynomial (called the *Pfaffian*) in the matrix elements with coefficients ± 1. Note that we need to fix a sign in the choice of pf; so insist $\mathrm{pf}(\psi_r) = 1$ for all r. For any $\alpha \in M_{2r}(A)$ and any alternating matrix $\varphi \in M_{2r}(A)$ we have $\mathrm{pf}(\alpha^t \varphi \alpha) = \mathrm{pf}(\varphi)\det(\alpha)$. For alternating matrices φ, ψ it is easy to check that $\mathrm{pf}(\varphi \perp \psi) = (\mathrm{pf}(\varphi))(\mathrm{pf}(\psi))$.

Two matrices $\alpha \in M_r(A)$, $\beta \in M_s(A)$ are said to be *equivalent* (w.r.t. EA) if there is a $\varepsilon \in E_{2(r+s+l)}(A)$, for some l, such that $\alpha \perp \psi_{s+l} = \varepsilon^{\mathrm{t}}(\beta \perp \psi_{s+l})\varepsilon$, (the t stands for 'transpose'). Denote this by $\alpha \underset{E}{\sim} \beta$. $\underset{E}{\sim}$ is an equivalence relation; denote by $[\alpha]$ the orbit of α under this relation. Moreover, a matrix equivalent to an alternating matrix is itself alternating and has the same Pfaffian.

It is easy to see (cf. [8, p. 945]) that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1; this group is called the Elementary Symplectic Witt group and is denoted by $\mathrm{W_E}A$.

(2.4) **M. Karoubi's theorem and square roots in** $\mathrm{W_E}(R[X])$. A famous theorem of M. Karoubi asserts that any invertible alternating matrix $V(X)$ over a polynomial ring $R[X]$ is stably congruent to its constant form if $1/2 \in R$, i.e. there is an l, and a $\sigma \in Sl_s(R[X])$, for suitable s, such that $\sigma^{\mathrm{t}}(V(X) \perp \psi_l)\sigma = V(0) \perp \psi_l$. The machination of M. Karoubi's proof (cf. [8, §3]) gives

(2.4.1) **Proposition.** *Let R be a local ring with $1/2k \in R$, and let $[V] \in \mathrm{W_E}(R[X])$. Then $[V]$ has a kth root, i.e. there is a $[W] \in \mathrm{W_E}(R[X])$ such that $[V] = [W]^k$ in $\mathrm{W_E}(R[X])$.*

Proof. Since R is local $\mathrm{W_E}(R) \equiv 0$, so we may assume that $V(0) = \psi_r$ for some r. Let me describe M. Karoubi's process showing V is stably congruent to $V(0)$; for details consult [8, §3]. The first step is to "stably make $V(X)$ linear" (known as the "Higman trick")—i.e. find an $\varepsilon \in E_{2(r+t)}(R[X])$ such that

$$\varepsilon^{\mathrm{t}}(V \perp \psi_t) = \psi_{r+t} + nX,$$

for some $t \geq 0$, some $n \in M_{2(r+t)}(R)$.

Since $\gamma = I_{r+t} - \psi_{r+t}nX \in Sl_{2(r+t)}(R[X])$, $\psi_{r+t}n$ is nilpotent, i.e. $(\psi_{r+t}n)^l \equiv 0$ for some l. Hence, if $1/2k \in R$, we can extract a kth root of γ $(= \beta^{2k}$ say) for some $\beta \in Sl_{2(r+t)}(R[X])$. Now M. Karoubi pointed out that

$$(*) \qquad \varepsilon^{\mathrm{t}}(V \perp \psi_t)\varepsilon = \psi_{r+t}\gamma = \psi_{r+t}\beta^{2k} = (\beta^k)^{\mathrm{t}}\psi_{r+t}\beta^k.$$

Let $W = \beta^{\mathrm{t}}\psi_{r+t}\beta$. Then applying Whitehead's lemma one can check that $W \perp W \perp \cdots \perp W$ (k times) $\underset{E}{\sim} V$, i.e. $[V] = [W]^k$ in $\mathrm{W_E}(R[X])$.

(2.5) **The antipodal vectors equality in** $\mathrm{Um}_3(R[X])$ **in small dimensions.** In [3, Lemma 1.3.1] we showed that if a $v = (v_0, v_1, \ldots, v_d) \in \mathrm{Um}_{d+1}(A)$, where A is essentially of dimension d, can be elementarily transformed to (its antipodal vector) $-v = (-v_0, v_1, \ldots, -v_d)$ then for all n, $[(v_0^n, v_1, \ldots, v_d)] = [v]^n$ in $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$. There are many examples of vectors which cannot be elementarily transformed to their antipodal vector; but in [3, §1.5] we showed that if $A = R[X]$, R a local ring of dimension 2 with $1/2 \in R$, then for any $v \in \mathrm{Um}_3(R[X])$, $v \underset{E}{\sim} -v$. Here, by a a different argument, we show that

(2.5.1) **Proposition.** *Let R be a local ring of dimension ≤ 4 with $1/2 \in R$ and let $v = (v_0, v_1, v_2) \in \mathrm{Um}_3(R[X])$. Then $v = (v_0, v_1, v_2) \underset{E}{\sim} (-v_0, -v_1, -v_2) = -v$.*

Proof. Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_2 + v_2w_2 = 1$, and consider the alternating matrix V with Pfaffian 1 given by

$$V(v, w) = \begin{Bmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{Bmatrix} \in Sl_4(R[X]).$$

Since $1/2 \in R$, by M. Karoubi's theorem (cf. §2.4) there is a

$$\beta \in Sl_{4+2l}(R[X]),$$

for some l, such that $\beta^t(V \perp \psi_l)\beta = \psi_{l+2}$. Since $\dim R \leq 4$, by [7, Theorem 2.6], $\mathrm{Um}_r(R[X]) = e_1E_r(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find a $\beta^* \in Sl_4(R[X])$ such that $(\beta^*)^t V\beta^* = \psi_2$.

Let $\delta =$ diagonal $(-1, 1, -1, 1) \in E_4(R)$. Then $\delta^t\psi_2\delta = -\psi_2$. Thus

$$(*) \qquad \begin{aligned} &\delta^t(\beta^*)^tV\beta^*\delta = \delta^t\psi_2\delta = -\psi_2 = \psi_2^t = [(\beta^*)^tV\beta^*]^t = (\beta^*)^tV^t\beta^*, \\ &\text{and so if } \sigma = (\beta^*)^t \text{ then } (\sigma^{-1}\delta^t\sigma)V(\sigma^{-1}\delta^t\sigma)^t = -V. \end{aligned}$$

By [7, Corollary 1.4] $\sigma^{-1}\delta^t\sigma \in E_4(R[X])$. Now the equation $(*)$ will prove the proposition on applying [11, Theorem 10].

(2.5.2) *Remark.* The above argument can be suitably modified to show that if $[V] \in \mathrm{W_E}(R[X])$, where R is a local ring with $1/2 \in R$, then $[V] = [-V]$ in $\mathrm{W_E}(R[X])$.

(2.6) **"Coordinate squares" in $\mathrm{W_E}(R[X])$.** Let us say that an invertible alternating matrix V is a "*coordinate kth power*" if the first row of V has the form $(0, v_1^k, v_2, \ldots, v_{2r-1})$. It would be of interest to know if, under congenial conditions, the above fact, proven in Proposition 2.4.1, that every $[V] \in \mathrm{W_E}(R[X])$ is a kth power in $\mathrm{W_E}(R[X])$ (under suitable hypothesis on R) can be translated to read that $[V]$ has a representative V^* which is a coordinate kth power and which, moreover, has the same size as that of V. We give some evidence for this here.

Firstly recall some multiplicative relations in $\mathrm{W_E}(A)$ observed by L. N. Vaserstein in [8, Theorem 5.2(a$_2$)].

(2.6.1) **The Vaserstein Rule.** *Let $v_1 = (a_0, a_1, a_2)$, $v_2 = (a_0, b_1, b_2)$ be unimodular vectors. Suppose that $a_0a_0' + a_1a_1' + a_2a_2' = 1$, and that*

$$v_3 = (a_0, (b_1, b_2)\left(\begin{smallmatrix} a_1 & a_2 \\ -a_2' & a_1' \end{smallmatrix}\right)) \in \mathrm{Um}_3(A).$$

Then for any w_1, w_2, w_3 such that $v_i \cdot w_i^t = 1$, $i = 1, 2, 3$, we have

$$[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \textit{in } \mathrm{W_E}(A).$$

(Note. $V(v, w)$ is defined in Proposition 2.5.1, and $[V(v, w)]$ is well defined in $\mathrm{W_E}(A)$ via [8, Lemma 5.1].)

(2.6.2) **Corollary.** (i) *Let* $v_1 = (a_0, a_1, a_2)$, $v_2 = (b_0, a_1, a_2)$ *be unimodular vectors. Suppose that* $a_0a_0' + a_1a_1' + a_2a_2' = 1$ *and that* $v_3 = (a_0(b_0 + a_0') - 1, (b_0 + a_0')a_1, a_2) \in \mathrm{Um}_3(A)$. *Then for any* w_1, w_2, w_3 *such that* $v_1 \cdot w_i^t = 1$, $i = 1, 2, 3$, *we have*

$$[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } \mathrm{W_E}(A).$$

(ii) *Let* $v_1 = (a_0, a_1, a_2)$, $v_2 = (b_0^2, a_1, a_2)$ *be unimodular vectors. Suppose that* $v_3 = (a_0b_0^2, a_1, a_2)$ *and that* w_1, w_2, w_3 *are such that* $v_iw_i^t = 1$, $i = 1, 2, 3$, *then*

$$[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } \mathrm{W_E}(A).$$

Proof. (i) is immediate from the Vaserstein Rule. We refer the reader to [9, Theorem 3.16(iii)] for deriving (ii) from (i). Note: You may need the Roitman lemma in [5, Lemma 1].

(2.6.3) **The "antipodal vectors equality" lemma in $\mathrm{W_E}(A)$.** *Let* $v = (v_0, v_1, v_2)$ *be a unimodular vector and assume that* $v \underset{E}{\sim} -v \underset{E}{\sim} (-v_0, -v_1, -v_2)$. *Let* $v_1^{(2)} = (v_0^2, v_1, v_2)$ *and let* w, w_1 *be such that* $v \cdot w^t = v^{(2)} \cdot w_1^t = 1$. *Then*

$$[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } \mathrm{W_E}(A).$$

Proof. Imitate the argument in [3, Lemma 1.3.1] in $\mathrm{W_E}(A)$. (Note. You will need Corollary 2.6.2(ii) above.)

Finally, we give some conditions under which we can extract "coordinate squares" in $\mathrm{W_E}(R[X])$;

(2.6.4) **Corollary.** *Let* R *be a local ring of dimension* ≤ 4 *with* $1/2 \in R$ *and let* $v = (v_0, v_1, v_2)$, $v^{(2)} = (v_0^2, v_1, v_2)$ *be unimodular* $R[X]$*-vectors. Let* w, w_1 *such that* $v \cdot w^t = v^{(2)} \cdot w_1^t = 1$. *Then,*

$$[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } \mathrm{W_E}(R[X]).$$

Proof. This will follow from Proposition 2.5.1 and Lemma 2.6.3.

(2.6.5) **Proposition.** *Let* R *be a local ring of dimension* ≤ 3 *with* $1/2 \in R$ *and let* $V \in Sl_4(R[X])$ *be an alternating matrix with Pfaffian* 1. *Then* $[V] = [V^*]$ *in* $\mathrm{W_E}(R[X])$ *with* $V^* \in Sl_4(R[X])$ *a coordinate square. Consequently, there is a stably elementary* $\gamma \in Sl_4(R[X])$ *such that* $V = \gamma^t V^* \gamma$.

Proof. By Proposition 2.4.1, $[V] = [W]^2$ for some $[W] \in \mathrm{W_E}(R[X])$. By [7, Theorem 2.6] $\mathrm{Um}_r(R[X]) = e_1E_r(R[X])$ for all $r \geq 5$, and so on applying [8, Lemma 5.3 and Lemma 5.5] a few times, if necessary, we can find an alternating matrix $W^* \in Sl_r(R[X])$ (with Pfaffian 1) such that $[W] = [W^*]$. Now apply Corollary 2.6.4 to find V^* as required. The last statement follows as above (only applying [8, Lemma 5.5 and Lemma 5.6] instead).

3. The main theorem

(3.1) **Theorem.** *Let R be a local ring of Krull dimension three with $1/2 \in R$ and let $v = (v_0, v_1, v_2)$ be a unimodular 3-vector over $R[X]$. Then v can be completed to an invertible matrix.*

Proof. Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_1 + v_2w_2 = 1$, and consider the alternating matrix V with Pfaffian 1 given by

$$V = \begin{Bmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{Bmatrix} \in Sl_4(R[X]).$$

Since $1/2 \in R$, by M. Karoubi's theorem (see $(*)$ in Proposition 2.4.1) there is a $\alpha \in Sl_{4+l}(R[X])$, for some l, such that $\alpha^{t}(V \perp \psi_l)\alpha = \psi_{l+2}$.

Since $\dim R = 3$, by [7, Theorem 2.6] $\mathrm{Um}_r(R[X]) = e_1E_r(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find an $\alpha \in Sl_4(R[X])$ such that $\alpha^{t}V\alpha = \psi_2$. Consider $e_4\alpha^{t}$, where $e_4 = (0, 0, 0, 1)$.

By [3, Proposition 1.4.4] $e_4\alpha^{t} \underset{E}{\sim} (a_0(X), a_1(X), a_2(X), c)$, where $c \in R$ is a non-zero-divisor in R. Let the 'overbar' denote 'modulo (c)'. By [3, Proposition 2.2], $(\overline{a_0(X)}, \overline{a_1(X)}, \overline{a_2(X)}) \underset{E}{\sim} (\overline{b_0(X)}^2, \overline{b_1(X)}, b_2(X))$, for some $b_0(X), b_1(X), b_2(X) \in R[X]$. On "lifting" this elementary map, and after an appropriate elementary transformation further, we can arrange that $e_4\alpha^{t} \underset{E}{\sim} (b_0(X)^2, b_1(X), b_2(X), c)$.

By Proposition 2.2.2, $(b_0(X), b_1(X), b_2(X), c)$ can be completed to an invertible matrix, say $\beta \in Sl_4(R[X])$ with $e_4\beta = (b_0(X), b_1(X), b_2(X), c)$.

Via Remark 1.1.1 follows that

$$\begin{aligned} e_4\beta^{-2}\alpha^{t} &= [e_4\beta^{-2}] * [e_4\alpha^{t}] = [e_4\beta]^{-2} * [e_4\alpha^{t}] \\ &= ([(b_0(X), b_1(X), b_2(X), c)]^2)^{-1} * [e_4\alpha^{t}] = [e_4\alpha^{t}]^{-1} * [e_4\alpha^{t}] \equiv 1, \end{aligned}$$

the last equality being deduced via [3, Example 1.5.3 and Lemma 1.3.1]. Thus, $\beta^{-2}\alpha^{t} = \varepsilon'\delta'$ for some $\varepsilon' \in E_4(R[X])$ and $\delta' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)$ with $\delta \in Sl_3(R[X])$.

Now $\psi_2 = \alpha^{t}V\alpha = (\beta^2\varepsilon'\delta')V(\beta^2\varepsilon'\delta')^{t} = \beta^2V^*(\beta^2)^{t}$, where $e_1V^* = (0, v\delta^{t}\varepsilon)$ for some $\varepsilon \in E_3(R[X])$—this will follow as $\delta' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)$ and via [11, Theorem 10].

By Proposition 2.6.5 there is a stably elementary $\gamma \in Sl_4(R[X])$ such that $\beta V^*\beta^{t} = \gamma^{t}V^{**}\gamma$, with $V^{**} \in Sl_4(R[X])$ a coordinate square. Let $e_1V^{**} = (0, a^2, b, c)$, and let α_0 (cf. §2.2) be a completion of (a^2, b, c).

Since

$$c_1V^{**} = e_1\begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}^{t}\psi_2\begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}$$

it follows via [8, Lemma 5.1] that

$$V^{**} = \varepsilon_1^{t}\begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}^{t}\psi_2\begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}\varepsilon_1$$

for some $\varepsilon_1 \in E_4(R[X])$. Thus,

$$\beta V^* \beta^{\mathrm{t}} = \gamma^{\mathrm{t}} V^{**} \gamma = \gamma^{\mathrm{t}} \varepsilon_1^{\mathrm{t}} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix}^{\mathrm{t}} \psi_2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} \varepsilon_1 \gamma .$$

Hence,

$$\beta^{-1} \left[\begin{pmatrix} 1 & 0 \\ 0 & \alpha_0^{-1} \end{pmatrix}^{\mathrm{t}} (\varepsilon_1^{-1})^{\mathrm{t}} (\gamma^{-1})^{\mathrm{t}} \right] \beta V^* \beta^{\mathrm{t}} \left[\gamma^{-1} \varepsilon_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0^{-1} \end{pmatrix} \right] (\beta^{-1})^{\mathrm{t}}$$
$$= \beta^{-1} \psi_2 (\beta^{-1})^{\mathrm{t}} = \beta^{-1} (\beta^2 V^* (\beta^2)^{\mathrm{t}}) (\beta^{-1})^{\mathrm{t}} = \beta V^* \beta^{\mathrm{t}} = \gamma^{\mathrm{t}} V^{**} \gamma ;$$

and so if

$$\theta = \beta^{\mathrm{t}} \gamma^{-1} \varepsilon_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} (\beta^{\mathrm{t}})^{-1} \gamma^{-1} , \text{ then } \theta^{\mathrm{t}} V^* \theta = V^{**} .$$

Compute $e_4 \theta^{\mathrm{t}}$ in the abelian group $\mathrm{Um}_4(R[X])/E_4(R[X])$ via Remark 2.1.1 to get $[e_4 \theta^{\mathrm{t}}] = [e_4 (\gamma^{\mathrm{t}})^{-1}]^2$. But γ is stably elementary and so via [3, Proposition 2.6] $[e_4 (\gamma^{\mathrm{t}})^{-1}]^2 = 1$; hence $[e_4 \theta^{\mathrm{t}}] = 1$, i.e. $e_4 \theta^{\mathrm{t}} \underset{E}{\sim} e_4$. Hence

$$\theta^{\mathrm{t}} \varepsilon' = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^{\mathrm{t}} \end{pmatrix}$$

for some $\theta' \in Sl_3(R[X])$, $\varepsilon' \in E_4(R[X])$.

Now

$$\theta^{\mathrm{t}} V^* \theta = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^{\mathrm{t}} \end{pmatrix} (\varepsilon')^{-1} V^* ((\varepsilon')^{-1})^{\mathrm{t}} \begin{pmatrix} 1 & 0 \\ 0 & \theta' \end{pmatrix} = V^{**} ,$$

and so via [11, Theorem 10] we can deduce that there is an $\varepsilon'' \in E_3(R[X])$ such that $v \varepsilon'' \theta' = (a^2, b, c)$. Since (a^2, b, c) is completable, so is v.

Remark. Let us, following M. Krusemeyer, say that a vector $v \in \mathrm{Um}_r(A)$ is *skew-completable* if there is an invertible alternating matrix $V \in Sl_{r+1}(A)$ with its first row $e_1 V = (0, v)$.

By making some appropriate modifications in the argument used to prove Theorem 3.1 one can show that,

(3.2) **Theorem.** *Let R be a local ring of Krull dimension d with $1/2 \in R$, and let $v = (v_0, v_1, \dots, v_{d-1})$ be a skew-completable vector over $R[X]$. Then v can be completed to an invertible matrix.*

Finally, using the well-known "Quillen-Suslin" Monic inversion and Local-Global principles, one can derive from $S_d(R)$ and Theorem 3.1 that,

(3.3) **Corollary.** *Let R be a noetherian ring of dimension 3 with $1/6 \in R$. Then any stably extended projective module over $R[X_1, \dots, X_n]$ is extended.*

Note added in proof. The contents (especially the mode of proof of the main result) of this note seems of interest in connection with the following problem:

(i) *Let $V : \mathrm{Um}_3(A)/E_3(A) \to \mathrm{W_E}(A)$ be the Vaserstein symbol. Is this map injective if* $\dim A = 3$?

I also hope that, after incorporation of some additional theories, the techniques used here will provide some insight towards settling,

(a) *Let* R *be a local ring with* $\frac{1}{2} \in R$. *Is every* $v \in \mathrm{Um}_3(R[X])$ *completable?*

(b) *Let* A *be a smooth affine algebra over the field* $\mathbf{C}$ *of complex numbers of dimension* d. *Is a stably free* A*-module of rank* $(d-1)$ *a free module?*

In an article entitled *On some actions of stably elementary matrices on alternating matrices* we prove that

"*Let* A *have Krull dimension* ≤ 5, *and let* $V \in Sl_4(A) \cap E_5(A)$ *be a stably elementary alternating matrix of Pfaffian one. Then* $V^8 \in E_4(A)$."

Note. One needs to show that $V \in E_4(A)$ to settle (i) above.

We also give some examples of 3 dimensional affine algebras for which the Vaserstein symbol V is bijective.

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